Differential Reduction Algorithms for the All-Order Epsilon Expansion of Hypergeometric Functions

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Hypergeometric functions provide a useful representation of Feynman diagrams occuring in precision phenomenology. In dimensional regularization, the ε-expansion of these functions about $d = 4$ is required. We discuss the current status of differential reduction algorithms. As an illustration, we consider the construction of the all-order ε-expansion of the Appell hypergeometric function F_1 around integer values of the parameters and present an explicit evaluation of the first few terms.

It is well-known that the hypergeometric representation is one of the most fruitful tools in the investigation of analytical properties of Feynman diagrams, and for efficiently evaluating these diagrams. The manipulation of hypergeometric functions can be separated into two distinct problems:

(A) deriving relations between hypergeometric functions with different parameters;

(B) finding an algorithm for the construction of the Laurent expansion of hypergeometric functions.

The first problem was solved by mathematicians [1], wheres the second one, which has been analyzed mainly by physicists, is still a subject of active research. The series of our recent papers has been devoted to the latter problem. [2, 3, 4] There are three different ways to describe special functions of the type occurring in Feynman diagrams:

(i) as an integral of the Euler or Mellin-Barnes type;

(ii) by a series whose coefficients satisfy certain recurrence relations;

(iii) as a solution of a system of differential and/or difference equations (holonomic approach).

For functions of a single variable, all of these representations are equivalent, but some properties of the function may be more evident in one representation than another. These three different representations have led physicists to three separate approaches to developing the ε-expansion of hypergeometric functions in Feynman diagram calculations.

The most impressive result in the Euler integral representation was the construction of the all-order ε-expansion of Gauss hypergeometric functions with special values of parameters in terms of Nielsen polylogarithms [5]. Such Gauss hypergeometric functions are related to one-loop propagator-type diagrams with arbitrary masses and momenta, two-loop bubble diagrams with arbitrary masses, and one-loop massless vertex-type diagrams.

The series representation has also been very useful and intensively studied. The first results of this type were derived in context of the so-called "single scale" diagrams [6]. Particularly impressive results involving series representations were derived in the framework of "nested sums" in Ref. [7] and in a generating function approach in Refs. [3, 8].

The differential equations satisfied by hypergeometric functions provide another approach. Results for Gauss hypergeometric functions expanded about integer and half-integer values of parameters were presented in Ref. [2, 9], and results for a special type of rational coefficients in Ref. [8], while results for generalized hypergeometric functions ${}_pF_{p-1}$ with integer values of parameters were presented in Ref. [4]. The iterated solution approach provides an efficient algorithm for constructing the ε-expansion, since each new term is related to previously known terms.

An important tool for the iterated solution is the iterated integral defined for each k by $I(a_k, a_{k-1}, \ldots, a_1; z) = \int_0^z \frac{dt}{t-a_k} I(a_{k-1}, \ldots, a_1; t)$. A special case of this integral,

$$G_{m_k,m_{k-1},\ldots,m_1}(a_k,\ldots,a_1;z) \equiv I(\underbrace{0,\ldots,0}_{m_k-1 \text{ times}}, a_k, \cdots, \underbrace{0,\ldots,0}_{m_1-1 \text{ times}}, a_1; z) \; ,$$

is related to multiple polylogarithms [10]

$$\mathrm{Li}_{k_1,k_2,\ldots,k_n}\left(x_1,x_2,\ldots,x_n\right) = \sum_{m_n>m_{n-1}>\cdots>m_2>m_1>0}^{\infty} \frac{x_1^{m_1}}{m_1^{k_1}}\frac{x_2^{m_2}}{m_2^{k_2}} \times \cdots \times \frac{x_n^{m_n}}{m_n^{k_n}} \;,$$

by

$$\begin{aligned} G_{m_n,\ldots,m_1}\left(x_n,\ldots,x_1;z\right) &= (-1)^n \mathrm{Li}_{m_1,m_2,\ldots,m_n}\left(\frac{x_2}{x_1},\frac{x_3}{x_2},\ldots,\frac{z}{x_n}\right) . \\ \mathrm{Li}_{k_1,\ldots,k_n}\left(y_1,\ldots,y_n\right) &= (-1)^n G_{k_n,\ldots,k_1}\left(\frac{1}{y_n},\ldots,\frac{1}{y_1\times\cdots\times y_n};1\right) , \end{aligned}$$

where we have used $I(a_1,\ldots,a_k;z) = I\left(\frac{a_1}{z},\ldots,\frac{a_k}{z};1\right)$.

As an illustration of our technique, we shall consider the Appell hypergeometric function F_1 defined by the series $F_1(a,b_1,b_2,c;z_1,z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}$. For this function, the differential reduction algorithm [1] is equivalent to the statement that any function $F_1(a,b_1,b_2,c;z_1,z_2)$ can be expressed as a linear combination of function with arguments that differ from the original ones by an integer, $F_1(a{+}m_0;b_1{+}m_1;b_2{+}m_2;c{+}m_3;z_1,z_2)$, and its first derivatives $R_3F_1(\vec{A}{+}\vec{m};z_1,z_2) = \left[\sum_{j=1,2}R_j\theta_j + R_0\right]F_1(\vec{A};z_1,z_2)$, where $\theta_r = z_r\frac{d}{dz_r}$, $r=1,2$, $\vec{A}=(a,b_1,b_2,c)$ is a list of parameters, the $\vec{m}$ are lists of integers, and the R_i are polynomials in parameters a,b_r,c and z_r. of basis functions The basis function $\omega_0(z_1,z_2) = F_1(a\varepsilon,b_1\varepsilon,b_2\varepsilon,1{+}c\varepsilon;z_1,z_2)$ satisfies the system of differential equations

$$\left[(1{-}z_1)\frac{\partial}{\partial z_1}\left(\theta_1{+}\theta_2\right)\right]\omega_0(z_1,z_2) = \left[\left(a{+}b_1{-}\frac{c}{z_1}\right)\varepsilon\theta_1{+}b_1\varepsilon\theta_2{+}ab_1\varepsilon^2\right]\omega_0(z_1,z_2) \;, \tag{1}$$

$$\left[(1{-}z_2)\frac{\partial}{\partial z_2}\left(\theta_1{+}\theta_2\right)\right]\omega_0(z_1,z_2) = \left[\left(a{+}b_2{-}\frac{c}{z_2}\right)\varepsilon\theta_2{+}b_2\varepsilon\theta_1{+}ab_2\varepsilon^2\right]\omega_0(z_1,z_2) \;. \tag{2}$$

Due to the analyticity of F_1 with respect to its parameters, Eqs. (1) – (2) hold for every coefficient function $\omega_0^{(k)}(z_1,z_2)$ in the expansion $\omega_0(z_1,z_2) = 1+\sum_{k=1}^{\infty}\omega_0^{(k)}(z_1,z_2)\varepsilon^k$, and the coefficient equations are

$$(1{-}z_1)\frac{\partial}{\partial z_1}\left(\omega_1^{(j)}(z_1,z_2){+}\omega_2^{(j)}(z_1,z_2)\right) = \left[a{+}b_1{-}\frac{c}{z_1}\right]\omega_1^{(j-1)}(z_1,z_2){+}b_1\omega_2^{(j-1)}(z_1,z_2){+}ab_1\omega_0^{(j-2)}(z_1,z_2) \;, \tag{3}$$

$$(1{-}z_2)\frac{\partial}{\partial z_2}\left(\omega_1^{(j)}(z_1,z_2){+}\omega_2^{(j)}(z_1,z_2)\right) = \left[a{+}b_2{-}\frac{c}{z_2}\right]\omega_2^{(j-1)}(z_1,z_2){+}b_2\omega_1^{(j-1)}(z_1,z_2){+}ab_2\omega_0^{(j-2)}(z_1,z_2) \;, \tag{4}$$

$$z_r\frac{\partial}{\partial z_r}\omega_0^{(j)}(z_1,z_2) = \omega_r^{(j)}(z_1,z_2) \;, r=1,2 \;, \tag{5}$$

where we have introduce new functions $\omega_r(z_1,z_2) = \theta_r\omega_0(z_1,z_2)$, $r=1,2$, which have ε-expansions of the form $\omega_r(z_1,z_2) = \sum_{k=1}^{\infty}\omega_r^{(k)}(z_1,z_2)\varepsilon^k$, $r=1,2$. To solve this system of first order differential equations, the boundary condition must be specified. Our choice is

$$\omega_r(z,z) = z\frac{ab_r\varepsilon^2}{1+c\varepsilon}\,{}_2F_1(1{+}a\varepsilon,1{+}(b_1+b_2)\varepsilon;2{+}c\varepsilon;z) \text{ and } \omega_0(z,z) = {}_2F_1(a\varepsilon,(b_1{+}b_2)\varepsilon;1{+}c\varepsilon;z),$$

where the all-order ε-expansion of the Gauss hypergeometric function has the form [4, 9]

$${}_2F_1(1{+}A\varepsilon,1{+}B\varepsilon;2{+}C\varepsilon;z) = \frac{1{+}C\varepsilon}{z}\sum_{j=0}^{\infty}\rho_j^{A,B,C}(z)\varepsilon^j \;, \quad {}_2F_1(A\varepsilon,B\varepsilon;1{+}C\varepsilon;z) = 1{+}AB\varepsilon^2\sum_{j=0}^{\infty}W_j^{A,B,C}(z)\varepsilon^j \;,$$

where $\rho_j^{A,B,C}(z)$ and $W_j^{a,b_1+b_2,c}(z)$ are expressible in terms of generalized polylogarithms. [10] In particular, for the coefficient functions $\omega_r^{(j)}(z_1,z_2)$, we have

$$\omega_1^{(j)}(z,z){+}\omega_2^{(j)}(z,z) = a(b_1{+}b_2)\rho_{j-2}^{a,b_1+b_2,c}(z) \;, \quad \omega_0^{(j)}(z,z) = a(b_1{+}b_2)W_{j-2}^{a,b_1+b_2,c}(z) \;.$$

The solution of Eqs. (3),(4) can now be written as

$$\omega_1^{(j)}(z_1,z_2){+}\omega_2^{(j)}(z_1,z_2) = \frac{1}{2}a(b_1{+}b_2)\left[\rho_{j-2}^{a,b_1+b_2,c}(z_1){+}\rho_{j-2}^{a,b_1+b_2,c}(z_2)\right]{+}\frac{1}{2}\int_{z_2}^{z_1}\frac{dt}{1-t}\left[f^{(j-1)}(t,z_2){-}h^{(j-1)}(z_1,t)\right],$$

which should be supplemented by a self-consistency condition,

$$\int_{z_2}^{z_1} \frac{dt}{1-t} \left[f^{(j-1)}(t,z_2) + h^{(j-1)}(z_1,t) \right] = a(b_1+b_2) \left[\rho_{j-2}^{a,b_1+b_2,c}(z_1) - \rho_{j-2}^{a,b_1+b_2,c}(z_2) \right],$$

where $f^{(j-1)}(z_1,z_2)$ and $h^{(j-1)}(z_1,z_2)$ are symbolic notation for the r.h.s. of Eqs. (3), (4), respectively. The last equation (5) may be rewritten as $\left(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right) \omega_0^{(j)}(z_1,z_2) = \omega_1^{(j)}(z_1,z_2) + \omega_2^{(j)}(z_1,z_2)$. The solution may be obtained by the method of characteristics, and we obtain a first-order ordinary differential equation depending on parameter $C = \frac{z_2}{z_1}$: $z_1 \frac{\partial}{\partial z_1} \omega_0^{(j)} = \omega_1^{(j)}(z_1, Cz_1) + \omega_2^{(j)}(z_1, Cz_1)$. Collecting all of these relations together provides an iterative algorithm for constructing the all-order ε-expansion of Appell hypergeometric function F_1 around integer values of the parameters. Due to the fact that all integral solutions contain only the factors $\frac{1}{1-t}$ and $\frac{1}{t}$, we could expect that the result of iteration will always be expressible in terms of a particular case of multiple polylogarithms.[10]

As an illustration, let us explicitly evaluate the first few coefficients. The first non-trivial term of the ε-expansion corresponds to $j = 2 (\omega_r^{(1)} = 0)$: $\omega_0^{(2)}(z_1,z_2) = \sum_{r=1,2} ab_r \, \mathrm{Li}_2(z_r)$ and $\omega_r^{(2)}(z_1,z_2) = -\sum_{r=1,2} ab_r \ln(1-z_r)$. The next iteration will produce

$$\begin{aligned} \frac{1}{a}\omega_0^{(3)}(z_1,z_2) \;=\; & \sum_{r=1,2} b_r \left[(a+b_r-c)\, \mathrm{S}_{1,2}(z_r) - c\, \mathrm{Li}_3(z_r) \right] + b_1 b_2 \left[\mathrm{S}_{1,2}(z_1) + \mathrm{S}_{1,2}(z_2) - \frac{1}{2} \ln^2\left(\frac{1-z_1}{1-z_2} \right) \ln z_1 \right] \\ & + b_1 b_2 \left\{ \left[\mathrm{Li}_2\left(\frac{z_2}{z_1} \frac{1-z_1}{1-z_2} \right) - \mathrm{Li}_2\left(\frac{1-z_1}{1-z_2} \right) \right] \ln\left(\frac{1-z_1}{1-z_2} \right) - \mathrm{Li}_3\left(\frac{z_2}{z_1} \frac{1-z_1}{1-z_2} \right) + \mathrm{Li}_3\left(\frac{1-z_1}{1-z_2} \right) \right\}. \end{aligned} \quad (6)$$

In summary, the hypergeometric function approach to Feynman diagrams provides a promising route to organizing and calculating the higher-order processes which are becoming increasingly important in precision phenomenology. Conversely, mathematicians may find that results motivated by high-energy physics will uncover new relationships among these classes of functions, providing a fertile area for interaction between mathematics and physics.

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